# Kuzmin-Osledets Formulations of Compressible Euler Equations

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**Abstract**

Kuzmin-Osledets formulations of compressible Euler equations for the barotropic case are considered. Exact results and physical interpretations are given. Symmetry restoration taking place at the Lagrangian level is pointed out.

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# 1 Introduction

Impulse formations of Euler (and Navier-Stokes) equations were considered by Kuzmin [1] and Osledets [2]. Different impulse formulations are produced by various possible gauge transformations (Russo and Smereka [3]). In the Kuzmin-Osledets gauge, the impulse variable $\mathbf{q}$ has an interesting geometrical meaning: it describes the evolution of material surfaces; its direction is orthogonal to the material surface element, and its length is proportional to the area of the surface element. The extension of the Kuzmin-Osledets formulation to the compressible barotropic case was considered in a brief way by Tur and Yanovsky [4]. In this paper, further details of this aspect are addressed - this includes exact results and physical interpretations.

# 2 Impulse Formulations of Compressible Euler Equations

Euler equations for a compressible fluid are

$$\frac{\partial \rho}{\partial t} + \triangledown \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

and

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \triangledown)\mathbf{v} = -\frac{1}{\rho}\triangledown p. \tag{2}$$

For a barotropic case, namely,

$$p = p(\rho) \tag{3}$$

equation (2) may be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\triangledown \times \mathbf{v}) = -\triangledown(P + \frac{1}{2}\mathbf{v}^2) \tag{4}$$

where

$$P(\rho) \equiv \int \frac{dp}{\rho}. \tag{5}$$

Introduce the Helmholtz decomposition -

$$\mathbf{q} = \mathbf{v} + \triangledown \phi \tag{6}$$

$\phi$ being an arbitrary scalar field; $\mathbf{q}$ then evolves, from equation (4), according to

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{v} \times (\triangledown \times \mathbf{q}) = -\triangledown(P + \frac{1}{2}\mathbf{v}^2 - \frac{\partial \phi}{\partial t}) \equiv \triangledown \psi. \tag{7}$$

In the Kuzmin-Osledets gauge, we take

$$\psi = -\mathbf{v} \cdot \mathbf{q} \tag{8}$$

which, from equation (7), implies the following gauge condition on $\phi$ :

$$\frac{\partial \phi}{\partial t} + (\mathbf{v} \cdot \triangledown)\phi + \frac{1}{2}\mathbf{v}^2 - P = 0. \tag{9}$$

Using (8), equation (7) becomes

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{v} \cdot \triangledown)\mathbf{q} = -(\triangledown \mathbf{v})^T \mathbf{q}. \tag{10}$$

On the other hand, taking the curl of equation (7) and using equation (1), we have for the evolution of vorticity -

$$\omega \equiv \bigtriangledown \times \mathbf{v} \tag{11}$$

the following equation for the generalized vorticity $\omega/\rho$ -

$$\frac{\partial}{\partial t}(\frac{\omega}{\rho}) + (\mathbf{v} \cdot \bigtriangledown)(\frac{\omega}{\rho}) = (\frac{\omega}{\rho} \cdot \bigtriangledown)\mathbf{v}. \tag{12}$$

# 3 General Results

**Theorem: q**, as defined in (6), satisfies the Kelvin-Helmholtz circulation theorem

$$\frac{d}{dt}\oint_C \mathbf{q} \cdot \mathbf{dl} = 0 \tag{13}$$

where C is a closed material curve in the fluid.

*Proof*: We have for a compressible barotropic fluid,

$$\frac{d}{dt}\oint_C \mathbf{v} \cdot \mathbf{dl} = 0 \tag{14}$$

which, on using (6), leads to

$$\frac{d}{dt}\oint_C \mathbf{q} \cdot \mathbf{dl} = 0 \quad or \quad \oint_C \mathbf{q} \cdot \mathbf{dl} = const \tag{13}$$

**Theorem**: The compressible barotropic flow has a Lagrange invariant -

$$[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \bigtriangledown)](\frac{\mathbf{q} \cdot \omega}{\rho}) = 0. \tag{15}$$

*Proof*: (15) follows immediately from equations (10) and (12).

(15) was given by Tur and Yanovsky [4], but the physical interpretation was not recognized which we develop in the following.

If **l** is a vector field associated with an infinitesimal line element of the fluid, **l** evolves according to (Batchelor [5]) -

$$[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \bigtriangledown)]\mathbf{l} = (\mathbf{l} \cdot \bigtriangledown)\mathbf{v} \tag{16}$$

which is identical to the potential vorticity equation (12). Therefore, the potential vortex lines evolve as fluid line elements.

On the other hand, if **S** is a vector field associated with an oriented material surface element of the fluid, **S** evolves according to (Batchelor [5])-

$$[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \bigtriangledown)](\rho\mathbf{S}) = -(\bigtriangledown\mathbf{v})^T(\rho\mathbf{S}) \tag{17}$$

which is identical to the equation of evolution of **q**, namely, equation (10). Therefore, the field lines of **q** evolve as fluid surface elements.

Thus, the Lagrange invariant

$$\frac{\mathbf{q} \cdot \omega}{\rho} = const \tag{18}$$

which may be called the potential helicity is simply physically equivalent to the mass conservation of the fluid element.[2]

On the other hand, the Lagrange invariant (18) also implies

$$\mathbf{q} \cdot \mathbf{l} = const \tag{19}$$

which may be seen to be a sufficient condition for the validity of the circulation conservation for $\mathbf{q}$ represented by (13).

Thus, the conservation laws of mass and momentum (and hence kinematics and dynamics) undergo a certain merger at the Lagrangian level signifying some symmetry restoration taking place there!

## 4 An Exact Solution

Consider the velocity field in $(r, \theta, z)$ coordinates -

$$\mathbf{v} =< V, \frac{\zeta}{r} U(\zeta), 0 > \tag{20}$$

where V is a constant and

$$\zeta \equiv r - Vt. \tag{21}$$

The vorticity associated with (20) is

$$\omega =< 0, 0, \frac{1}{r}[\zeta U(\zeta)]' > \tag{22}$$

Using (20), the mass conservation equation (1) leads to

$$\frac{\partial}{\partial t}(r\rho) + V \frac{\partial}{\partial r}(r\rho) = 0 \tag{23}$$

from which

$$\rho = \frac{1}{r} g(\zeta). \tag{24}$$

The incompressible limit

$$V = 0 \quad : \quad \rho = const \tag{25}$$

on application to (24), leads to

$$g(\zeta) = a\zeta \tag{26}$$

$a$ being an arbitrary constant.

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[2]For the stratifed flow case, for which equation (1) is replaced by

$$\bigtriangledown \cdot \mathbf{v} = 0 \quad and \quad \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \bigtriangledown \rho = 0$$

Ertel's invariant

$$\frac{\omega \cdot \bigtriangledown \rho}{\rho} = constant$$

is readily seen to be the appropriate version of (18). This also implies that Ertel's invariant, like (18), is physically equivalent to the mass conservation of the fluid element!

Using (26), (24) leads to

$$\rho = \frac{a}{r}\zeta. \tag{27}$$

Here, the parameters a and V have to be chosen appropriately so as to keep $\rho$ positive definite.

Next, equation (10) leads to

$$\frac{\partial q_r}{\partial t} + V\frac{\partial q_r}{\partial r} - (\frac{\zeta}{r}U)\frac{q_\theta}{r} = -\frac{q_\theta}{r}\frac{\partial}{\partial r}(\zeta U) - \frac{q_r V}{r} \tag{28}$$

$$\frac{\partial}{\partial t}(rq_\theta) + V\frac{\partial}{\partial r}(rq_\theta) = 0. \tag{29}$$

Equation (29) yields,

$$rq_\theta = f(\zeta). \tag{30}$$

Now, the equi-vorticity condition associated with (6), namely,

$$\bigtriangledown \times q = \bigtriangledown \times \mathbf{v} \tag{31}$$

gives,

$$q_\theta = \frac{\zeta}{r}U(\zeta). \tag{32}$$

Comparing (32) with (30), we have

$$f(\zeta) = \zeta U(\zeta). \tag{33}$$

Using (30) and (33), equation (28) becomes

$$\frac{\partial}{\partial t}(rq_r) + V\frac{\partial}{\partial r}(rq_r) = -\frac{\zeta^2}{r}UU' - \frac{\zeta Vt}{r^2}U^2. \tag{34}$$

On putting,

$$rq_r = \frac{\zeta^2}{r}G(\zeta)t \tag{35}$$

equation (34) leads to

$$\zeta G - \frac{\zeta Vt}{r}G = -\zeta UU' - \frac{Vt}{r}U^2 \tag{36}$$

from which,

$$G = -UU' \tag{37}$$

$$\zeta U' + U = 0. \tag{38}$$

(37) and (38) yield -

$$U(\zeta) = \frac{b}{\zeta} \tag{39}$$

$$G(\zeta) = \frac{b^2}{\zeta^3} \tag{40}$$

$b$ being an arbitrary constant. Using (39) and (40), we obtain from (32) and (35) -

$$q_r = \frac{b^2 t}{r^2\zeta} \quad , \quad q_\theta = \frac{b}{r}, \tag{41}$$

while (20) becomes

$$\mathbf{v} =< V, \frac{b}{r}, 0 > . \tag{42}$$

Thus, the flow under consideration is a density wave on a steady flow with corresponding $\mathbf{q}$ growing monotonically with $t$. On the other hand, (42) implies that this flow has zero vorticity -

$$\bigtriangledown \times \mathbf{v} = \bigtriangledown \times \mathbf{q} = \mathbf{0} \tag{43}$$

and hence has the trivial Clebsch representation:

$$\mathbf{q} = \psi \bigtriangledown \phi \tag{44}$$

with

$$\psi = const = 1, \quad \phi = \frac{b^2}{V^2 t} ln(\frac{r - Vt}{r}) + \frac{b^2}{Vr}. \tag{45}$$

## 5 Discussion

In this paper, Kuzmin-Osledets formulations of compressible Euler equations for the barotropic case are considered. The kinematics and dynamics aspects apparently undergo a certain unification at the Lagrangian level. These symmetries break as one moves up to the Eulerian level. The Kuzmin-Osledets formulation in the compressible case admits an exact solution that describes a density wave on a steady irrotational flow with corresponding $\mathbf{q}$ growing monotonically with $t$.

## 6 References


[1] G.A. Kuzmin: Phys Lett. A **96**, 88 (1983).
[2] V.I. Osledets: Russ. Math. Surveys **44**, 210 (1989).
[3] G. Russo and P. Smereka: J. Fluid Mech. **391**, 189 (1999).
[4] A.V. Tur and V.V. Yanovsky: J. Fluid Mech. **248**, 67 (1993).
[5] G.K. Batchelor: *An Introduction to Fluid Dynamics*, Cambridge Univ. Press, (1967).